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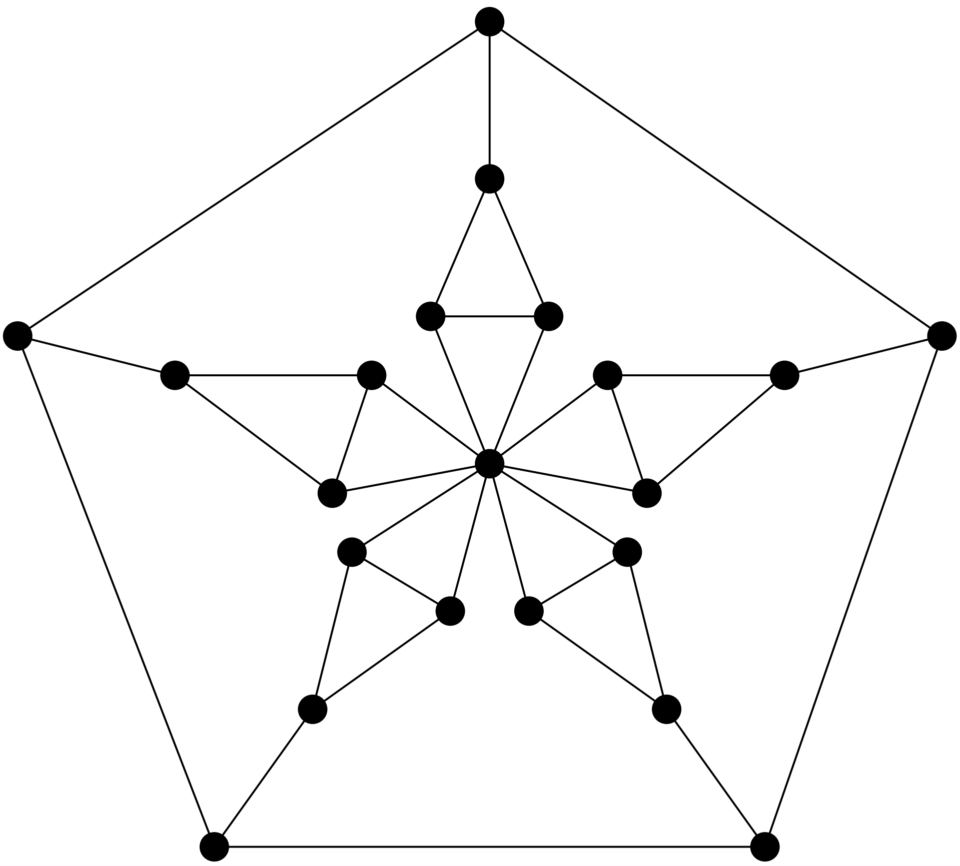
"Cycles, Colourings, Cliques, Claws and Closures"



SELECTED PROBLEMS

http://www.mathe.tu-freiberg.de/inst/theomath/rindex.php?seite=rathen

1 Minimum order of $k-$chromatic $K_{r+1}-$free graphs

(Anja Kohl)

Let $n_r(k)$ denote the smallest possible number of vertices of a graph G with chromatic number $\chi(G) = k$ and clique number $\omega(G) \leqslant r$. Obviously, $n_r(k) = k$ for $r \geqslant k$. So we are only interested in the case $r \leqslant k-1$.

It is known that $n_2(k)$ has order of magnitude $k^2 \cdot \log k$.

Question 1.1. *What order of magnitude has $n_r(k)$ for $r \geqslant 3$?*

For $k \leqslant 3+r$ all exact values for $n_r(k)$ are known, in particular: $n_2(4) = 11$ [1], $n_2(5) = 22$, $n_3(5) = 11$ [2] and $n_3(6) = 16$ [4].

Problem 1.1. *Determine $n_2(6)$, $n_3(7)$, and $n_4(8)$.*
(*known:* $30 \leqslant n_2(6) \leqslant 45$, $20 \leqslant n_3(7) \leqslant 31$, $17 \leqslant n_4(8) \leqslant 22$)

Known ([3]): For k sufficient large and $r \geqslant k-11$, we have the following results:

- $n_{k-i}(k) = k+2i$ for $k \geqslant 3i$, $i = 1, 2$
- $n_{k-i}(k) = k+2i$ for $k \geqslant 2i+1$, $i = 3, 4, 5, 6$
- $n_{k-i}(k) = k+2i-1$ for $k \geqslant 2i$, $i = 7, 8$
- $n_{k-i}(k) = k+2i-2$ for $k \geqslant 2i-1$, $i = 9$
- $n_{k-i}(k) = k+2i-3$ for $k \geqslant 2i-2$, $i = 10, 11$
- $n_{k-i}(k) = k+2i-4$ for $k \geqslant 2i-3$, $i = 13$.

Problem 1.2. *Determine $n_{k-12}(k)$ for $k \geqslant 22$.*
(*known for $k \geqslant 22$:* $k+20 \leqslant n_{k-12}(k) \leqslant k+21$).

The value of $n_{k-12}(k)$ for $k \geqslant 22$ depends on the Ramsey number $R(10,3)$. It is known that $40 \leqslant R(10,3) \leqslant 43$. If $R(10,3) \in \{42, 43\}$, then $n_{k-12}(k) = k+20$ for $k \geqslant 22$, otherwise $n_{k-12}(k) = k+21$.

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2 CHROMATIC NUMBER OF GRAPHS WITH TWO ODD CYCLE LENGTHS

(Stephan Matos Camacho)

Let $C_o(G)$ denote the set of odd cycle lengths contained by a graph G. Gyárfás ([2]) proved that G is colourable with at most $2k+1$ colours if $|C_o(G)| = k$, unless G contains a K_{2k+2}. In [1] we observed

Theorem 2.1. *If* G *only contains two consecutive odd lengths greater than 3, say* $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geqslant 3\}$*, then* G *is* 4*-colourable.*

Question 2.1. *Let* G *be a* 2*-connected graph with* $C_o(G) = \{2m+i : i \in \{-1,1\}, m \geqslant 3\}$ *and minimal degree at least 3. Is* $\chi(G) < 4$*?*

In [5] a first part of the question could be answered:

Theorem 2.2. *If* G *contains only odd cycles of length* 5 *and 7, then* G *is* 3*-colourable.*

The proof is based on known results on the structure of k-critical graphs due to Dirac and Toft. The rest of the questions still remains open.

Question 2.2. *Let* G *be a* $2-$*connected graph with* $C_o(G) = \{2m+1, 2n+1 : m > n+1, n \geqslant 2\}$ *and minimal degree at least 3. Is* $\chi(G) < 5$*?*

References:

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3 b-COLOURINGS OF GRAPHS

(Mais Alkhateeb, Anja Kohl)

Let G be a simple and undirected graph of order n. A b-*colouring* of G is a proper vertex colouring such that there is a vertex in each colour class, which is adjacent to at least one vertex in every other colour class. Such a vertex is called a *colour-dominating vertex*. The b-*chromatic number* of a graph G, denoted by $b(G)$, is the largest k such that there is a b-colouring of G by k colours.

Irving and Manlove ([3]) introduced the concept of b-colouring and proved that determining the $b-$chromatic number is NP-hard in general and polynomial for trees. Determining $b(G)$ is NP-hard even for bipartite graphs ([4]). Moreover, [3] states that $b(G) \leqslant m(G)$ for every graph G with m-degree $m(G) := \max\{1 \leqslant i \leqslant n : d(v_i) \geqslant i-1\}$ where the vertices $v_1, v_2, ..., v_n$ of G are ordered in nonincreasing order of their degrees.

Problem 3.1. *Characterization of graphs with* $m(G) - 1 \leqslant b(G) \leqslant m(G)$.

It is known, that this inequality holds for trees ([3]), and for graphs whose blocks are cliques of size 2 or 3.

Another interesting problem is to characterize those graphs G satisfying $b(G) = \Delta(G) + 1$. If this problem is limited to regular graphs, Kratochvíl et al. ([4]) proved that for every d-regular graph with at least d^4 vertices, $b(G) = d + 1$. Moreover, it is known that for every graph G with girth at least 6, $b(G) \geqslant \delta(G)$, and if this graph is d-regular then $b(G) = d + 1$.

Conjecture 3.1. *([4])] For every* d*-regular graph* G *with girth at least* 5, $b(G) = d+1$.

A graph G is called b-*continuous* if it has a b-colouring by k colours for all k satisfying $\chi(G) \leqslant k \leqslant b(G)$. There exist graphs which are not b-continuous, e.g. the cube Q_3, every $(r-1)$-factor of the complete bipartite graph $K_{r,r}$, $r \geqslant 4$ ([3]), and some other bipartite graphs ([2]). Since all non-b-continuous graphs that are known so far contain a claw as an induced subgraph, we ask:

Question 3.1. *Does there exist a claw-free graph that is not* b*-continuous?*

We conjecture that there is no such graph. This is reason to pose the following conjecture:

Conjecture 3.2. *Line graphs are* b*-continuous.*

Recently we could verify that graphs G with minimum degree $\delta(G) \geqslant n-3$ are b-continuous ([1]). Moreover, there exist non-b-continuous graphs G of order n and minimum degree $\delta(G) = n-5$. So we ask:

Question 3.2. *Is every graph G with minimum degree $\delta(G) = n-4$ b-continuous?*

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4 ON THE CYCLIC CHROMATIC NUMBER OF 3-CONNECTED PLANE GRAPHS

(Mirko Horňák)

The *cyclic chromatic number* of a plane graph G, in symbol $\chi_c(G)$, is a minimum number of colours in such a vertex colouring of G that distinct vertices incident with a common face receive distinct colours. If G is 2-connected, then $\chi_c(G) \geqslant \Delta^*(G)$, where $\Delta^*(G)$ is the maximum face degree of G.
On the other hand, no 3-connected plane graph G is known with $\chi_c(G) > \Delta^*(G)+2$. Plummer and Toft ([7]) proved that $\chi_c(G) \leqslant \Delta^*(G) + 9$ and conjectured (PTC) that $\chi_c(G) \leqslant \Delta^*(G) + 2$ for any 3-connected plane graph G.
Let $\mathrm{PTC}(d)$ denote PTC restricted to 3-connected plane graphs G with $\Delta^*(G) = d$. It is known that $\mathrm{PTC}(d)$ is true for $d = 3$ (Four Colour Theorem), $d = 4$ (Borodin [1]), $d \in \{18, \dots, 23\}$ (Horňák and Zlámalová [6]) and $d \geqslant 24$ (Horňák and Jendrol' [5]). For $\Delta^*(G) \geqslant 60$ Enomoto et al. ([4]) obtained the best possible inequality: $\chi_c(G) \leqslant \Delta^*(G) + 1$ (graphs of pyramids show that the bound $\Delta^*(G) + 1$ cannot be improved).
The best general upper bound known so far is due to Enomoto and Horňák ([3]), namely $\chi_c(G) \leqslant \Delta^*(G) + 5$.

Problem 4.1. *Prove* $\mathrm{PTC}(d)$ *for some* $d \in \{5, \dots, 17\}$.

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5 THE CIRCULAR TOTAL CHROMATIC NUMBER

(Andrea Hackmann, Arnfried Kemnitz)

A $k-$total colouring of a simple graph G is an assignment of k colours to the vertices and edges of G such that the neighbored elements - two adjacent vertices or two adjacent edges or a vertex incident to an edge - are coloured differently. The minimum number k for which a graph G admits a $k-$total colouring is the total chromatic number $\chi''(G)$ of G.

If k and d are positive integers with $k \geqslant 2d$ then a $(k,d)-$total colouring of a graph G is an assignment c of colours $\{0, 1, \ldots, k-1\}$ to the vertices and edges of G such that $d \leqslant |c(x_i) - c(x_j)| \leqslant k - d$ whenever two elements x_i and x_j are neighbored. *The circular total chromatic number* $\chi''_c(G)$ of G is defined as the infimum of fractions $\frac{k}{d}$ for all $(k,d)-$total colourings of G:

$$\chi''_c(G) = \inf\left\{\frac{k}{d} : \text{G has a } (k,d) - \text{total colouring.}\right\}$$

Obviously, a $(k,1)-$total colouring is a $k-$total colouring of G which implies that $\chi''_c(G) \leqslant \chi''(G)$. For cycles C_p it holds $\chi''_c(C_{3k+1}) = 3 + \frac{1}{k}$ and $\chi''_c(C_{3k+2}) = 3 + \frac{1}{2k+1}$ whereas $\chi''(C_{3k+1}) = \chi''(C_{3k+2}) = 4$.

For example, for complete graphs and several classes of complete multipartite graphs the total chromatic number and the circular total chromatic number coincide.

Problem 5.1. *Determine classes of graphs* G *aside from cycles such that*

$$\chi''_c(G) < \chi''(G).$$

6 CHOICE NUMBER OF CARTESIAN PRODUCTS

(Mieczysław Borowiecki, Stanislav Jendrol')

Let $\mathrm{ch}(G)$ denote the choice number of G and let $G \times H$ be the Cartesian Product of graphs G and H.
Galvin (1995) proved that $\mathrm{ch}(K_n \times K_n) = n$ and solved in this way the old Dinitz's conjecture. (See Diestel's book where the solution is presented as a result on the edge list colouring of bipartite multigraphs.)

Let G and H be graphs. Clearly,

$$max\{\mathrm{ch}(G), \mathrm{ch}(H)\} \leqslant \mathrm{ch}(G \times H).$$

Question 6.1. *Does there is an absolute constant* c *such that*

$$\mathrm{ch}(G \times H) \leqslant max\{\mathrm{ch}(G), \mathrm{ch}(H)\} + c?$$

Question 6.2. *If the answer is YES, then how big is* c*? Is* $c = 1$*?*

Comment: In [1] we have shown that the above conjecture is not true in a general case.
However, we suspect that the following two conjectures hold:

Conjecture 6.1. *There exists a constant* A *such that the following holds for every pair of graphs* G *and* H*:*

$$\mathrm{ch}(G \times H) \leqslant A(\mathrm{ch}(G) + \mathrm{ch}(H)).$$

Conjecture 6.2. *Let* G *and* H *be two graphs with maximum degree at most* Δ*. Then*

$$\mathrm{ch}(G \times H) \leqslant \Delta + o(\Delta).$$

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7 COLOURING VERTICES OF PLANE GRAPHS UNDER RESTRICTIONS GIVEN BY FACES

(Stanislav Jendrol')

Consider a vertex colouring of a connected plane graph G. A colour c is used k times by a face α of G if it appears k times along the facial walk of α. Two natural problems arise.

1. A vertex colouring φ is a *weak parity vertex colouring* of a connected plane graph G with respect to its faces if each face of G uses at least one colour an odd number of times. Problem is to determine the minimum number $\chi_w(G)$ of colours used in a wpv colouring of G.

In [1] it is proved that $\chi_w(G) \leqslant 4$ for every connected plane graph G with minimum degree at least 3. We strongly believe that the following holds.

Conjecture 7.1. *Let G be a connected plane graph of minimum face degree at least 3. Then*

$$\chi_w(G) \leqslant 3.$$

The Conjecture is true for 2-connected cubic plane graphs, see [1].

2. A vertex colouring φ is a *strong parity vertex colouring* of a 2-connected plane graph G with respect to the faces of G if each face of G that uses a colour then it uses an odd number of times. Problem is to find the minimum number $\chi_s(G)$ of colours used in an spv colouring of G. We believe that

Conjecture 7.2. *There is a constant k such that for every 2-connected plane graph G*

$$\chi_s(G) \leqslant k.$$

We do not know any 2-connected plane graph H with $\chi_s(H) \geqslant 7$. Hence, we believe that $k = 6$ in the above conjecture.

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8 ON TOTAL ACYCLIC COLOURING OF PLANAR GARPHS

Mieczysław Borowiecki, Izak Broere, Peter Mihók

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. A $(\mathcal{P}, \mathcal{Q})$-*total colouring* of a simple graph G is a colouring of the vertices $V(G)$ and edges $E(G)$ of G such that for each colour i the vertices coloured by i induce a subgraph of property $\mathcal{P}$, the edges coloured by j induce a subgraph of property $\mathcal{Q}$ and moreover the adjacent vertices and edges obtain different colours. The minimum number of colours of a total $(\mathcal{P}, \mathcal{Q})$-colouring of G is the *total* $(\mathcal{P}, \mathcal{Q})$-*chromatic number* and is denoted by $\chi''_{\mathcal{P},\mathcal{Q}}(G)$.
We will present basic results and some Problems, and Conjectures on total $(\mathcal{P}, \mathcal{Q})$-colourings of planar graphs where the class $\mathcal{P}$ and $\mathcal{Q}$ is the class of edgeless and acyclic graphs, respectively.
An *acyclic* k-*colouring* of a graph G is a proper vertex k-colouring of G satisfying the subgraph induced by **every pair of colour classes** has no cycle.
The minimum k such that G has an acyclic k-colouring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$.

Total acyclic 3-colouring

Theorem 8.1. *If a graph G is properly acyclic vertex k-colourable, then G is totally $(\mathcal{O}, \mathcal{D}_1)$-colourable with k colours when k is odd and with $k+1$ colours when k is even.*

Let G be a graph. Then $\chi''_{\mathcal{O},\mathcal{D}_1}(G) = 1$ if and only if G is an edgeless graph with at least one vertex. If G contains at least one edge, then $\chi''_{\mathcal{O},\mathcal{D}_1}(G) \geqslant 3$. Thus there is no graph G with $\chi''_{\mathcal{O},\mathcal{D}_1}(G) = 2$.
By Theorem 8.1 we have two Corollaries.

Corollary 1. *Let G be a forest with at least one edge. Then $\chi''_{\mathcal{O},\mathcal{D}_1}(G) = 3$ if and only if $\chi_a(G) = 2$.*

Corollary 2. *Let G be a graph with at least one cycle. Then $\chi''_{\mathcal{O},\mathcal{D}_1}(G) = 3$ if and only if $\chi_a(G) = 3$.*

Planar graphs

Theorem 8.2. ([1]) *Every planar graph has an acyclic 5-colouring.*

Theorems 8.1 and 8.2 imply the following result.

Corollary 3. *If G is a planar graph, then $\chi''_{\mathcal{O},\mathcal{D}_1}(G) \leqslant 5$.*

This upper bound also holds for larger class of graphs:

Theorem 8.3. *If* G *is* K_5*-minor free, then* $\chi''_{\mathcal{O},\mathcal{D}_1}(G) \leqslant 5$.

Theorem 8.4. (*[4]*) *If* G *is a planar bipartite graph, then* $\chi_a(G) \leqslant 5$ *and the bound is sharp.*

Although planar graphs and bipartite planar graphs have the same upper bound for the acyclic chromatic number, but for the graphs from the second class, we have the following theorem.

Theorem 8.5. *If* G *is a planar bipartite graph, then* $\chi''_{\mathcal{O},\mathcal{D}_1}(G) \leqslant 4$ *and the bound is sharp.*

Consider now the class of 3-colourable planar graphs. Apparently for these graphs we have the same upper bound, namely:

Theorem 8.6. *If* G *is a* 3*-colourable planar graph, then* $\chi''_{\mathcal{O},\mathcal{D}_1}(G) \leqslant 4$ *and the bound is sharp.*

Since for K_5-minor free graphs we have the same upper bound as for planar graphs it seems that Four Colour Theorem can be strengthened to the following one:

Conjecture 8.1. *If* G *is a planar graph, then* $\chi''_{\mathcal{O},\mathcal{D}_1}(G) \leqslant 4$.

We recently proved the following Conjecture:

Conjecture 8.2. (*[3]*) *If* G *is a planar graph, then* $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leqslant 4$.

We conclude with the following Problem:

Problem 8.1. *Characterize planar bipartite graphs with* $\chi''_{\mathcal{O},\mathcal{D}_1}(G) = 4$ *for which* $\chi_a(G) = 4$ $(\chi_a(G) = 5)$.

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9 List colourings of integer distance graphs

(Arnfried Kemnitz)

Let D be a subset of the positive integers $\mathbb{N}$. The *integer distance graph* $G(\mathbb{Z}, D) = G(D)$ is defined as the graph with the set of integers as vertex set, $V(G(D)) = \mathbb{Z}$, and edge set consisting of all pairs uv whose distance $|u - v|$ is an element of the so-called *distance set* D.

General bounds for the chromatic number of integer distance graphs are

$$2 \leqslant \chi(G(D)) \leqslant |D| + 1.$$

Voigt ([4]) and Zhu ([5]) determined $\chi(G(D))$ if $|D| = 3$:
If $D = \{x, y, z\}$ consists of integers whose greatest common divisor equals 1, then $\chi(D) = 4$ if and only if $D = \{1, 2, 3n\}$ or $D = \{x, y, x + y\}$ and $x \not\equiv y \pmod 3$. If x, y, z are odd then $\chi(D) = 2$. For all other 3−element distance sets D it holds $\chi(D) = 3$.

General bounds for the list chromatic number (choice number) of integer distance graphs are $\chi(D) \leqslant ch(D) \leqslant |D| + 1$ (Kemnitz, Marangio 2001).

Question 9.1. *Does there exist a* 3−*element distance set such that* $ch(D) < 4$*?*

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10 $[1,1,t]$-Colourings of Complete Graphs

Arnfried Kemnitz, Massimiliano Marangio

Given non-negative integers r, s, and t, an $[r,s,t]$-*colouring* of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the colour set $\{0, 1, \dots, k-1\}$ such that $|c(v) - c(v')| \geqslant r$ for every two adjacent vertices v, v', $|c(e) - c(e')| \geqslant s$ for every two adjacent edges e, e', and $|c(v) - c(e)| \geqslant t$ for all pairs of incident vertices and edges, respectively. The $[r,s,t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r,s,t]$-colouring.

This is an obvious generalization of all classical graph colourings since c is a vertex colouring if $r = 1$, $s = t = 0$, an edge colouring if $s = 1$, $r = t = 0$, and a total colouring if $r = s = t = 1$, respectively. Therefore, $\chi_{1,0,0}(G) = \chi(G)$, $\chi_{0,1,0}(G) = \chi'(G)$, and $\chi_{1,1,1}(G) = \chi''(G)$ where $\chi(G)$ is the chromatic number, $\chi'(G)$ the chromatic index, and $\chi''(G)$ the total chromatic number of the graph G.

For complete graphs K_n on n vertices it holds

$$\chi_{1,1,1}(K_n) = \chi''(K_n) = \begin{cases} n & \text{if } n \text{ odd,} \\ n+1 & \text{if } n \text{ even} \end{cases}$$

and we proved (see [1])

$$\chi_{1,1,2}(K_n) = \begin{cases} n & \text{if } n = 1, \\ n+2 & \text{if } n \geqslant 3 \text{ odd, } n = 2,\ n = 6,\ \text{or } n = 8, \\ n+3 & \text{if } n = 4 \text{ or } n \geqslant 10 \text{ even.} \end{cases}$$

Problem 10.1. *Determine* $\chi_{1,1,t}(K_n)$ *for* $3 \leqslant t \leqslant 2(n-1)$.

Note that $\chi_{1,1,t}(K_n) = 2(n-1) + t$ if $t > 2(n-1)$ (see [2]).

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11 NEIGHBOURS DISTINGUISHING INDEX OF PLANAR GRAPHS

(Keith Edwards, Mirko Horňák, Mariusz Woźniak)

Let G be a finite simple graph with no component K_2. Let C be a finite set of colours and let $\varphi : E(G) \to C$ be a proper edge colouring of G. The *colour set* of a vertex $v \in V(G)$ with respect to φ, in symbols $S_\varphi(v)$, is the set of colours of edges incident with v. The colouring φ is *neighbours distinguishing* if $S_\varphi(x) \neq S_\varphi(y)$ for any $xy \in E(G)$. For example, any neighbour-distinguishing colouring of C_5 uses necessarily 5 colours.
The *neighbours distinguishing index* of the graph Gis the smallest number $\mathrm{ndi}(G)$ of colours in a neighbour-distinguishing colouring of G. Neighbours distinguishing index has been introduced in [7], where the authors have conjectured that $\mathrm{ndi}(G) \leqslant \Delta(G)+2$ for any connected graph G nonisomorphic to C_5 on at least three vertices (Neighbour-Distinguishing Conjecture = NDC). NDC was confirmed in [1] for cubic graphs and for bipartite graphs, in [6] for graphs with maximum degree at most 3, in [2] for planar graphs with girth at least 6 and in [5] for planar graphs with maximum degree at least 12. In [3] it was proved that $\mathrm{ndi}(G) \leqslant \Delta(G)+1$ for any planar bipartite graph G with $\Delta(G) \geqslant 12$. Hatami in [4] showed that $\mathrm{ndi}(G) \leqslant \Delta(G)+300$ provided that $\Delta(G) > 10^{20}$.

Problem 11.1. *Find the minimum integer* $\Delta \geqslant 4$ *such that* $\mathrm{ndi}(G) \leqslant \Delta(G)+1$ *for any plane bipartite graph* G *with* $\Delta(G) \geqslant \Delta$.

Problem 11.2. *Prove or disprove NDC for planar graphs* G *with* $\Delta(G) = \Delta$ *for* (*at least some*) $\Delta \in \{4, \dots, 11\}$.

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12 EDGE-DISTINGUISHING INDEX OF A SUM OF CYCLES

(Rafał Kalinowski, Mariusz Woźniak)

A *neighbourhood* $N(e)$ of an edge e of a graph $G = (V, E)$ is a subgraph of G induced by e and all edges adjacent to e. A colouring $c : E \to S$ is called *edge-distinguishing* if, for any two distinct edges e, e', there does not exist an isomorphism φ of $N(e)$ onto $N(e')$ preserving colours of c, and such that $\varphi(e) = e'$. An *edge-distinguishing index* $\chi'_e(G)$ of a graph G is the minimum number of colours in a proper edge-distinguishing colouring $c : E \to S$.
If G is a cycle C_n of length n, then it is easy to see that

$$\chi'_e(C_n) \geqslant \gamma_n := \min\{ k \mid \frac{1}{2}k^2(k-1) \geqslant n \}.$$

Theorem 12.1.

$$\chi'_e(C_n) = \begin{cases} \gamma_n + 1 & \textit{if } n = \frac{1}{2}k^2(k-1) - 1 \textit{ or } n = 4, \\ \gamma_n & \textit{otherwise.} \end{cases}$$

Problem 12.1. *Let* G *be a disjoint sum of cycles with the total sum of lengths equal to* n. *Evaluate* $\chi'_e(G)$.

13 RAINBOW CONNECTION

(Ingo Schiermeyer)

An edge coloured graph is called *rainbow connected* if any two vertices are connected by a path whose edges have different colours. The rainbow connection number $rc(G)$ is the smallest number of colours that are needed in order to make G rainbow connected. It is known that

$$1 \leqslant rc(G) \leqslant n-1.$$

Problem 13.1. *For every* $k \geqslant 2$ *find a minimal constant* c_k *with* $0 < c_k \leqslant 1$ *such that* $rc(G) \leqslant c_k \cdot n$ *for all graphs* G *with minimum degree* $\delta(G) \geqslant k$*. Is it true that*

$$c_k = \frac{3}{k+1} \text{ for all } k \geqslant 2?$$

This is true for $k = 2, 3$ ($c_2 = 1, c_3 = \frac{3}{4}$).

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14 SATURATED RAINBOW EDGE COLOURING OF CUBE GRAPHS

(Heiko Harborth, Arnfried Kemnitz)

Let $f(n,k)$ denote the minimum number of colours for the edges of the cube graph Q_n such that for $k < n$ no rainbow Q_k occurs (the edges of a rainbow Q_k have pairwise different colours), however, for every edge with a colour used at least twice it follows that a new colour for this edge induces a rainbow Q_k.
For $k = 3$ it is known $f(3,3) = 11$, $f(4,3) = 24$, and $f(5,3) = 20$.

1. Determine $f(6,3)$.
2. Determine the smallest $n > 3$ such that $f(n,3) = 11$.

15 TIGHTNESS OF BONDY'S THEOREM

(Arnfried Kemnitz)

Bondy([1]) proved in 1980 that a k-connected graph G of order $n \geqslant 3$ is hamiltonian if the minimum degree sum $\sigma_{k+1}(G)$ of all $(k+1)$-sets of independent vertices is at least $\frac{1}{2}((k+1)(n-1)+1)$ which generalizes the sufficient conditions of Dirac $(k=0)$ and Ore $(k=1)$.

Bondy's bound is tight for $k=1$ and all n since for $G \cong (K_{n-2} \cup K_1) + K_1$ it holds that $\sigma_2(G) = \lceil \frac{1}{2}(2(n-1)+1) \rceil - 1 = n-1$ but G is not hamiltonian.

The bound is also tight for all k and odd n since the graph $G \cong K_{\frac{n-1}{2}} + \frac{n+1}{2} K_1$ fulfills $\sigma_{k+1}(G) = \lceil \frac{1}{2}((k+1)(n-1)+1) \rceil - 1$ but is not hamiltonian.

Question: Is Bondy's bound also tight if n is even?

Note that for $k=2$ and $n=8$ the bound $\sigma_3(G) \geqslant 11$ is not tight since all 927 non-hamiltonian 2-connected graphs G of order 8 fulfill $\sigma_3(G) \leqslant 9$.

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16 Every locally connected graph is weakly pancyclic

(Zdenek Ryjáček)

Let G be a finite simple undirected graph and let $g(G)$ and $c(G)$ be the girth and the circumference of G (i.e. the length of a shortest cycle of G and the length of a longest cycle of G), respectively. We say that G is *weakly pancyclic* if G contains cycles of all lengths ℓ for $g(G) \leqslant \ell \leqslant c(G)$. The graph G is *locally connected* if the neighborhood of every vertex of G induces a connected graph.

Conjecture 16.1. ([8]) *Every connected locally connected graph is weakly pancyclic.*

Comments. The concept of locally connected graphs was introduced by Chartrand and Pippert ([3]). More information about weakly pancyclic graphs appears in [2], for example.

The conjecture is based on a result by Clark ([4]), who proved that every connected, locally connected graph is vertex pancyclic (having cycles of all lengths from 3 to $|V(G)|$ through every vertex). Without the claw-free assumption, it is easy to construct locally connected graphs that are nonhamiltonian. Nevertheless, all known examples are weakly pancyclic; and indeed [4] proved the conjecture for claw-free graphs.

In a chordal graph, every block is locally connected, and for every cycle of length at least 4 there is a cycle with length one less that is obtained by skipping one vertex. Thus the conjecture holds for chordal graphs.

It is easy to show that the square of any graph is locally connected. (The *square* adds edges making vertices at distance 2 in the original graph adjacent.) Fleischner ([5], Theorem 6) proved that the square of every graph is weakly pancyclic, thus verifying the conjecture for squares of graphs.

The lexicographical product of graphs is another way to obtain a locally connected graph. Kaiser and Kriesell ([6]) recently proved that the lexicographical product $G[H]$ is weakly pancyclic provided G is a connected graph and H is an arbitrary graph with at least one edge.

Kriesell ([7]) verified the conjecture for graphs with maximum degree at most 4.

Finally, planar triangulations are locally connected. Balister ([1]) proved the conjecture for this class as follows. Let C be a cycle in a planar triangulation G. By induction on the number of faces inside, we prove that the interior (with boundary) contains cycles of all shorter lengths. If some face inside has two edges on C, then using the third edge yields a cycle C' with length one less and fewer faces inside. Otherwise, there is a face with one edge on C and the third vertex inside. Detouring from C to include this vertex forms a longer cycle C', but again it has fewer regions inside and the induction hypothesis applies.

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17 Dominating cycles and hamiltonian prisms

(Zdenek Ryjáček)

The *prism over a graph* G, denoted $G \square K_2$, is the Cartesian product of G and K_2. It consists of two disjoint copies of G and a perfect matching connecting a vertex in one copy of G to its “clone" in the other copy.

A graph G is *hamiltonian* if it has a hamiltonian cycle and *traceable* if it has a hamiltonian path. Define a k-*walk* in a graph to be a spanning closed walk in which every vertex is visited at most k times.

The following implications are easy to verify:

$$\begin{aligned} G \text{ is hamiltonian} \Rightarrow G \text{ is traceable} \Rightarrow G \square K_2 \text{ is hamiltonian} \\ \Rightarrow G \text{ has a 2-walk.} \end{aligned}$$

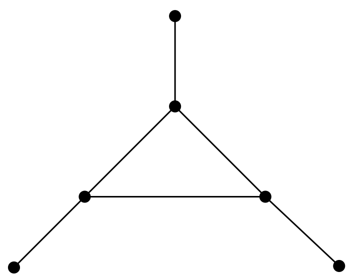
Thus the question whether $G \square K_2$ is hamiltonian is “sandwiched” between hamiltonicity and having a 2-walk. Specifically, the property of having a hamiltonian prism can be considered as a “relaxation" of hamiltonicity. More information about prism-hamiltonicity of a graph can be found e.g. in [1] and [2].

A *dominating cycle* in a graph G is a cycle C such that every edge of G has at least one vertex on C, i.e. such that the graph $G - C$ is edgeless. Clearly, a hamiltonian cycle is dominating, and hence the property of having a dominating cycle can be considered as another relaxation of hamiltonicity.

There is a natural question whether there is any relation between these two properties.

Example 17.1. Let H be any 2-connected cubic nonhamiltonian graph, and let G be obtained from H by replacing every vertex of H with a triangle (such a G is sometimes called the *inflation* of H). Then G is a 2-connected line graph and these are known [2] to be prism-hamiltonian. On the other hand, since H is nonhamiltonian, any cycle in G has to miss at least one “new" triangle and hence G has no dominating cycle. Thus, there are “many" graphs showing that hamiltonian prism does not imply having a dominating cycle.

Example 17.2. The graph in the figure below shows that also the existence of a dominating cycle does not imply having hamiltonian prism.



However, all such known examples are of low toughness. This motivates the following question.

Conjecture 17.1. *Let* G *be a 1-tough graph having a dominating cycle. Then* G *has hamiltonian prism.*

Comments. Recall that G is 1-tough if, for any $S \subset V(G)$, the graph $G - S$ has at most $|S|$ components.

Suppose that G has a dominating cycle C of even length. Set $M = V(G) \setminus V(C)$ and $N = \{x \in V(C) | \ x \text{ has a neighbor in } M\}$. Then the graph induced by $M \cup N$ has a matching containing all vertices from M (this follows by the toughness assumption and by the Hall's theorem). Using this matching, it is easy to construct a hamiltonian cycle in $G \square K_2$.

The difficult case is when all dominating cycles in G are of odd length.

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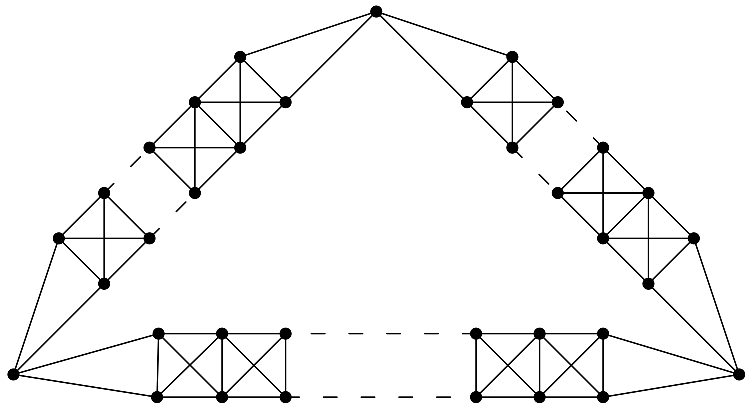
18 NONPANCYLIC CLAW-FREE GRAPHS WITH COMPLETE CLOSURE

(Zdenek Ryjáček, Richard Schelp)

It is known that a claw-free graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian. On the other hand, there are nonpancyclic graphs with pancyclic closure [1]. The graph in the figure below is an example of such a nonpancyclic graph with complete (and hence pancyclic) closure.



Problem 18.1. *Determine the maximum number of cycle lengths that can be missing in a claw-free graph on n vertices with complete closure.*

It is easy to see that a claw-free graph with complete closure on at least 4 vertices can miss neither a C_3 nor a C_4. The main result of [2] shows that such a graph G cannot be missing a cycle of length $n-1$; however, the proof of this result is difficult and cannot be iterated.

The following was conjectured in [2].

Conjecture 18.1. *Let c_1, c_2 be fixed constants. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, where $3 \leqslant i \leqslant c_1$ and $n - c_2 \leqslant i \leqslant n$.*

Recently, counterexamples to the first part of the Conjecture 18.1 have been found (see [3]), all these counterexamples have connectivity $\kappa \leqslant 5$. We believe that the second part of Conjecture 18.1 is true, and that such a construction as shown in [3] is possible only for connectivity $\kappa \leqslant 5$. Thus, we conjecture the following.

Conjecture 18.2. *Let c be a fixed constand. then for large n, any claw-free graph G of order n, whose closure is complete, contains cycles C_i for all $i, n - c \leqslant i \leqslant n$.*

Conjecture 18.3. *Every 6-connected claw-free graph with complete closure is pancyclic.*

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19 CYCLES CONTAINING $k-$CONNECTED VERTEX SUBSETS AND INDEPENDENT EDGE SUBSETS

(Jochen Harant)

Given $k \in \mathbb{N}$, a graph G, and $X \subseteq V(G)$. X is k-*connected in* G if $G - S$ has a component containing X for every $S \subseteq V(G)$ with $|S| \leqslant k - 1$.

Remark: Even in a planar graph this 'regional' connectedness of a vertex set X can be arbitrarily large (in opposition to the usual global connectedness of a planar graph, i.e. if $X = V(G)$)!

Theorem 19.1. *Let* G *be a planar graph,* $X \subseteq V(G)$*, such that* X *is* 4*-connected in* G*, and* $E \subset E(G[X])$*, such that* $|E| \leqslant 2$ *and* E *is independent if* $|E| = 2$.
Then there is a cycle of G *containing* $X \cup E$.

Theorem 19.2. *Let* G *be a graph,* $X \subseteq V(G)$*, such that* $|X| \geqslant 7$ *and* X *is* $(|X| - 3)$*-connected in* G*, and* $E \subset E(G[X])$*, such that* $|E| = 3$ *and* E *is independent.*
Then there is a cycle of G *containing* $X \cup E$.
(Not true if X *is only* $(|X| - 4)$*-connected in* G*.)*

Problem 19.1. *Is there a positive constant* c *such that the following holds ?*
A planar graph G *with* $X \subseteq V(G)$*,* X *is* c*-connected in* G*,* $E \subset E(G[X])$*,* $|E| = 3$*, and* E *is independent, has a cycle containing* $X \cup E$.

20 L VERTICES ON A SHORT CYCLE

(Jochen Harant)

Let G be a k–connected graph of order $n = |V(G)|$. Given l prescribed vertices, $1 \leqslant l \leqslant k$, find a short cycle containing these l vertices. By a theorem of Dirac such a cycle always exists. Denote the length of this cycle in the worst case by $f(n, k, l)$.

Question 20.1.

$$f(n, k, k) = \frac{2}{k}n + c_k \ ?$$

Known:

1. $f(n, k, 1) = \frac{2}{\binom{k}{2}}n + \text{const}$
2. $f(n, k, 2) = f(n, k, 3) = \frac{2}{k}n + \text{const}$
3. For $l > k$: If such a cycle exists, then $f(n, 3, 4) \geqslant \frac{3}{4}n + \text{const}$

21 HAMILTONIAN NEIGHBORHOOD GRAPHS

(Martin Sonntag, Hanns-Martin Teichert)

The *neighborhood graph* $N(G)$ for a simple graph $G = (V, E)$ is defined to be the graph on the same vertex set V with two vertices adjacent if and only if there is in G a path of length two between them. Neighborhood graphs, also referred to as *two-step graphs*, have been the object of several studies in the last 25 years.

If G is hamiltonian then $N(G)$ is hamiltonian for |V| odd. This is not true for |V| even, for instance $N(C_{2n})$ is disconnected. We can show that $N(G)$ is always hamiltonian if G is 1-hamiltonian connected and has a triangle, but we think there are weaker conditions providing hamiltonicity of $N(G)$.

Problem 21.1. *Find sufficient conditions for* G, *such that* $N(G)$ *is hamiltonian or hamiltonian connected.*

22 ARBITRARILY VERTEX-DECOMPOSABLE TREES

(Mirko Horňák, Antoni Marczyk and Mariusz Woźniak)

A tree T is said to be *arbitrarily vertex decomposable* if for any sequence $(t_1, \ldots, t_k)$ of positive integers adding up to $|V(T)|$ there is a sequence $(T_1, \ldots, T_k)$ of vertex-disjoint subtrees of T such that $|V(T_i)| = t_i$ for $i = 1, \ldots, k$.
The notion of an arbitrarily vertex decomposable (avd for short) tree has been introduced independently by Barth et al. in [1] and Horňák and Woźniak in [5].

It turned out that some classes are essential when analysing the property of a tree "to be avd". A *star-like tree* (a *spider*) is a tree homeomorphic to a star $K_{1,q}$. Such a tree is uniquely (up to isomorphism) determined by the non-decreasing sequence $(a_1, \ldots, a_q)$ of orders of its arms; it will be denoted by $S(a_1, \ldots, a_q)$ and also called a q-*spider*.
A *caterpillar* is a tree T having as a subgraph a path P such that $T - P$ is an edgeless graph.

The most general result concerns the best upper bound avd_{max} on the maximum degree of an avd tree. In [5] it has been conjectured that $\text{avd}_{\text{max}} \leqslant 6$ and conjectured that $\text{avd}_{\text{max}} = 4$. Later it was shown that $\text{avd}_{\text{max}} \leqslant 5$ ([7]) and $\text{avd}_{\text{max}} \leqslant 4$ ([2]). More precisely, the result of [2] reads as follows:

Theorem 22.1. *If a tree* T *is avd, then* $\Delta(T) \leqslant 4$. *Moreover, if a tree* T *is avd, then each vertex of* T *of degree four is adjacent to a leaf.*

Let us mention that there are avd trees with maximum degree 4, for example $S(2, 2, 5, 7)$, hence, as conjectured, $\text{avd}_{\text{max}} = 4$.
There are only few known families of avd trees. The following theorem has been proved independently in [1] and [5] (see also [4] for another, much more complicated result of this type).

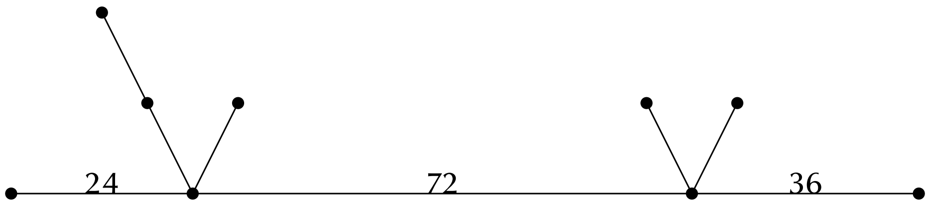
Theorem 22.2. *A* 3*-spider* $S(2, a_2, a_3)$ *is avd if and only if* a_2 *and* a_3 *are coprime.*

For $a_1 \geqslant 2$ let $A_2(a_1)$ be the set of all a_2's such that $a_2 \geqslant a_1$ and there is $a_3 \geqslant a_2$ such that $S(a_1, a_2, a_3)$ is avd. Similarly, for $a_2 \geqslant a_1 \geqslant 2$ let $A_3(a_1, a_2)$ be the set of all a_3's such that $a_3 \geqslant a_2$ and there is $a_4 \geqslant a_3$ such that $S(a_1, a_2, a_3, a_4)$ is avd.
From Theorem 22.1 it is clear that if $A_3(a_1, a_2) \neq \emptyset$, then $a_1 = 2$.
We give below four "main" open questions concerning avd trees.

Question 22.1. *Is* $A_2(a_1) \neq \emptyset$ *for all* $a_1 \geqslant 2$*?*

Question 22.2. *Is* $A_3(2, a_2) \neq \emptyset$ *for all* $a_2 \geqslant 2$*?*

Horňák and Woźniak ([6]) showed that $A_2(a_1) \neq \emptyset$ for all $a_1 \in \{2, \ldots, 28\}$ and $A_3(2, a_2) \neq \emptyset$ for all $a_2 \in \{2, \ldots, 23\}$. According to [3] there are infinitely many a_1's such that $A_2(a_1) \neq \emptyset$.
It is easy to see that an avd caterpillar has at most one vertex of degree four. In Figure 1 there is depicted an avd tree having two vertices of degree four.



An edge labelled l stands for a subpath with l vertices of degree 2.

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23 WHICH GRAPH INVARIANTS ARE ARHP?

(Peter Mihók, Gabriel Semanišin)

Let $\varphi(G)$ be a graph invariant. A graph G is called φ-*partitionable* if, for any pair of positive integers (k_1, k_2) satisfying $k_1 + k_2 \geqslant \varphi(G) - 1$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $\varphi(G[V_1]) \leqslant k_1$ and $\varphi(G[V_2]) \leqslant k_2$.
The next well-known results for the maximum degree $\Delta(G)$ provides an illustration of such an invariant:

Theorem 23.1. (*Lovász, 1966*) *Every graph* G *is* Δ*-partitionable.*

The following problem have been formulated by I.Schiermeyer during the workshop Hereditarnia 2003 (see [1]):

Question 23.1. *Which other partition concepts/problems of this type do exist?*

We have investigated a problem related to the previous one: Let $\mathcal{P}$ be a hereditary property of graphs. Given a graph invariant φ, we define the associated *invariant of the property* $\mathcal{P}$ in the following manner:

$$\varphi(\mathcal{P}) = \min\{\varphi(F) : F \in \mathbf{F}(\mathcal{P})\}.$$

The motivation for the investigation of invariants related to hereditary graph properties comes from extremal and chromatic graph theory. The classical Erdős-Stone-Simonovits formula provides a relationship between the maximum number of edges in a $\mathcal{P}$-maximal graph of order n and the invariant $\chi(\mathcal{P})$ - the chromatic number of $\mathcal{P}$ (see e.g. [7]).

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be any properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. The property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. If a property $\mathcal{R}$ can be expressed as the product of at least two properties,then it is said to be *reducible*; otherwise it is called *irreducible* (for more details see e.g. [3]).

A. Berger ([2]) proved that any reducible additive hereditary property of graphs has infinitely many minimal forbidden graphs. But only very little is known about the structure of $\mathbf{F}(\mathcal{P} \circ \mathcal{Q})$, even in the case when the structure of $\mathbf{F}(\mathcal{P})$ and $\mathbf{F}(\mathcal{Q})$ is known. Moreover, A. Farrugia proved in [4], that recognizing whether a graph belongs to a property $\mathcal{P} \circ \mathcal{Q}$ (i.e. recognizing whether it contains a graph from $\mathbf{F}(\mathcal{P} \circ \mathcal{Q})$ as a subgraph) is polynomial only in the simplest case : if the property $\mathcal{P} \circ \mathcal{Q}$ is the property "to be bipartite". Useful information on the structure of $\mathbf{F}(\mathcal{P} \circ \mathcal{Q})$ can be obtained by investigation of graph invariants associated with the property $\mathcal{P}\circ\mathcal{Q}$.

We say that a graph invariant φ is *additive with respect to reducible hereditary properties* (abbreviated by ARHP) if for any reducible property $\mathcal{P}\circ\mathcal{Q}$ the equality $\varphi(\mathcal{P}\circ\mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ is valid.

In [5] we proved that the subchromatic number $\psi(G) = \chi(G) - 1$ is ARHP and we stated the following problem:

Question 23.2. *Which graph invariants are ARHP?*

In [6] we presented a necessary and sufficient condition for a graph invariant to be ARHP and we proved that amongst the others the degeneracy number and tree-width are ARHP. Our investigation stimulates us to formulate the following more specific problem:

Question 23.3. *Is the choice number* $\mathrm{ch}(G)$ *ARHP?*

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24 VERTEX PARTITIONS AND GRAPH POLYNOMIALS

(Peter Tittmann)

Let $G = (V, E)$ be an undirected finite graph with n vertices. A partition $\pi \in \mathbb{P}(V)$ of the vertex set of G is called *independent* if all blocks of π are independent vertex sets of G. Here $\mathbb{P}(V)$ denotes the set of all partitions of V. Let a_i be the number of independent partitions with i blocks of G. Then the *chromatic polynomial* of G is

$$P(G, x) = \sum_{i=0}^{n} a_i x^{\underline{i}},$$

where $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$ denotes the falling factorial of x. A *connected partition* $\pi \in \mathbb{P}(V)$ consists of blocks that induce connected subgraphs of G. The set $\mathbb{P}_c(G)$ of all connected partitions of G forms a geometric sublattice of $\mathbb{P}(V)$ ([4]). Rota ([3]) showed that the chromatic polynomial can be obtained from $\mathbb{P}_c(G)$:

$$P(G, x) = \sum_{\sigma \in \mathbb{P}_C(G)} \mu\left(\hat{0}, \sigma\right) x^{|\pi|}$$

Let $q_i(G)$ be the number of connected partitions of $G = (V, E)$ with exactly i blocks and

$$Q(G, x) = \sum_{i=0}^{n} q_i(G)\, x^i.$$

We call $Q(G, x)$ the *partition polynomial* of G. Two graphs with coinciding partition polynomial are said to be *Q-equivalent*. We can derive from $Q(G, x)$ the number of edges, vertices, triangles, components, and the girth of G. For a vertex $v \in V$, we denote by $N(v)$ the set of vertices that are adjacent to v in G. Let $X \subseteq V$ and let G_X the graph obtained from $G = (V, E)$ by merging all vertices of X into a single vertex. Possibly arising parallel edges are replaced by single edges. Then the following equation is valid for each vertex $v \in G$:

$$Q(G, x) = xQ(G - v, x) + \sum_{\emptyset \subset W \subseteq N(v)} (-1)^{|W|+1} Q(G_W, x)$$

Problem 24.1. *The lattice $\mathbb{P}_c(G)$ of connected partitions determines the chromatic polynomial uniquely. In which way does the partition polynomial (which may be considered as rank generating function of $\mathbb{P}_c(G)$) determine the chromatic polynomial? Are there Q-equivalent graphs with different chromatic polynomials?*

Problem 24.2. *The chromatic polynomial and the adjoint polynomial ([2]) can be computed in polynomial time for graphs of bounded treewidth. Can we find an* *algorithm that computes the partition polynomial of a graph of bounded treewidth in polynomial time?*

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25 VERTEX-DISJOINT INDEPENDENT SETS

(Anja Kohl)

While investigating the minimum order of k–chromatic K_{r+1}–free graphs the following question concerning Ramsey numbers arises:

Question 25.1. *Let* G *be a graph with clique number at most* r*, independence number 3, and order* $n = R(r+1,3)+1$*. Do there always exist two vertex-disjoint independent sets with 3 vertices?*

Known: If the order of G is $n = R(r+1,3)+2$, then the answer is "yes".

26 DOMINATION HYPERGRAPHS OF TOURNAMENTS

(Martin Sonntag, Hanns-Martin Teichert)

Let $D = (V, A)$ be a digraph. A subset $V' \subseteq V$ is called a *dominating set* iff $\forall x \in V \setminus V' \; \exists \, y \in V' : (y, x) \in A$. The *domination graph* $\mathcal{D}(D)$ has vertex set V and its edges are the dominating sets of cardinality two (see for instance [1]). As a natural generalization the *domination hypergraph* $\mathcal{DH}(D)$ also has vertex set V and its edges are all minimal dominating sets V' with $|V'| \geqslant 1$.
There are many interesting results on domination graphs of tournaments T_n, e.g. in general $\mathcal{D}(T_n)$ is not connected (see for instance [2]).

Conjecture 26.1. *The domination hypergraph $\mathcal{DH}(T_n)$ of a tournament T_n is always connected.*

The conjecture is true for $n \leqslant 9$. We tested hundreds of bigger examples (up to $n = 23$) by *Mathematica* routines and found no counterexample. It is easy to prove that every nontrivial component of $\mathcal{DH}(T_n)$ contains at least three edges. A first step to verify the conjecture could be the investigation of regular tournaments.

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27 Weight of graphs having a given property

(Stanislav Jendrol')

Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}_{1+}$ denote bipartite graphs, connected graphs and graphs of minimum degree at least 1. The following results have been proved in [1].

Theorem 27.1. *If* $a^* = \left\lceil \frac{n-\sqrt{n^2-4m}}{2} \right\rceil$, $b^* = \lceil \frac{m}{a^*} \rceil$ *and* $p^* = \min(a^*b^* - m, 2)$, *then* $a^* + b^* - p^* \leqslant W(n, m, \mathcal{B}) \leqslant a^* + b^* - p^* + 1$.

There are sufficient conditions for $W(n, m, \mathcal{B}) = a^* + b^* - p^*$:

- $p^* \in \{0, 1\}$
- $b^* \geqslant 2a^* + 1$
- $n \geqslant 40$ and $m \leqslant \lfloor \frac{2n^2-6n}{9} \rfloor$

A necessary condition for $W(n, m, \mathcal{B}) = a^* + b^* - p^* + 1$: there is $l \in \mathbb{Z}^+$ such that $m = (a^* + l)(b^* - l - 1)$.

Let k, m', r, c, d, e be integers defined by $\binom{n}{2} - \binom{k+1}{2} < m \leqslant \binom{n}{2} - \binom{k}{2}$, $m' := \binom{n}{2} - \binom{k}{2} - m$, $r := \lceil \frac{m'}{n-k} \rceil$, $c := 1$ if either $m' \leqslant \lfloor \frac{n-k}{2} \rfloor$ or $m' = (n-k-1)^2$, $c := 2$ otherwise, $d := \left\lfloor \frac{2m}{n} \right\rfloor$, $e := 0$ if either $m = \binom{n}{2} - 1$ or $d \leqslant n - 3$ and $2m \equiv q \pmod{n}$, $0 \leqslant q \leqslant d - 1$, and $e := 1$ otherwise.

Theorem 27.2. *If* $48 \leqslant n - 1 \leqslant m \leqslant \binom{n}{2} - \binom{\lceil n/2 \rceil}{2}$, $k \geqslant \frac{n}{2}$ *and* $\mathcal{C} \subseteq \mathcal{P} \subseteq \mathcal{D}_{1+}$, *then* $W(n, m, \mathcal{P}) = 2n - k - r - c$.

Theorem 27.3. *If* $n \geqslant 15$, $\binom{n}{2} - \binom{\lceil n/2 \rceil}{2} + 1 \leqslant m \leqslant \binom{n}{2}$ *and* $\mathcal{C} \subseteq \mathcal{P} \subseteq \mathcal{D}_{1+}$, *then* $W(n, m, \mathcal{P}) = 2d + e$.

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28 FRACTIONAL GRAPH PROPERTIES

(Peter Mihók)

A graph property $\mathcal{P}$ is any nonempty isomorphism-closed class of simple (finite or infinite) graphs. We will consider additive and hereditary graph properties i.e. classes closed under disjoint union and subgraphs.
The fractional invariants were introduced in [2]. In 1999 we considered with Zsolt Tuza and Margit Voigt (see [1]) the generalized fractional chromatic number of graph. Here we stay some problems which are related to some generalized invariants.
Let a, b be positive integers, $a > b$ and $\mathcal{P}$ be an additive and hereditary graph property. A *fractional* (*circular*) $(\mathcal{P}, a : b)$-colouring of a graph G is a mapping ϕ of $V(G)$ to the set of all b-element subsets (of b consecutive elements modulo a) of $\{0, 1, \dots, a-1\}$ such that for each "'colour"' $i, 0 \leqslant i \leqslant a-1$ the subgraph $G[i]$ induced by the vertices where $i \in \phi(v)$ has the property $\mathcal{P}$.
For a given property $\mathcal{P}$ the class of all graphs which possess a *fractional* and *circular* $(\mathcal{P}, a : b)$-colouring will be denoted by $\mathcal{P}^{a:b}$ and $\mathcal{P}^{[a:b]}$.
For given different graph properties $\mathcal{P}, \mathcal{Q}$ we say that $\mathcal{P}$ is a bound for $\mathcal{Q}$ if $\mathcal{Q} \subset \mathcal{P}$.
Let us consider the properties:

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \text{ the maximum degree } \Delta(G) \leqslant k\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e., the minimum degree } \delta(H) \leqslant k,\\
&\qquad \text{for each } H \subseteq G\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2}\\
&\qquad \text{or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},
\end{aligned}$$

It is known that the class $\mathcal{T}_3$ of planar graphs has the bounds $\mathcal{O}^{[4:1]}$; $\mathcal{D}_1^{[3:1]}$; $(\mathcal{D}_1 \cap \mathcal{S}_2)^{[3:1]}$; $\mathcal{D}_1^{[5:2]}$.
We propose to study best fractional and circular bounds for given graph properties in the following sense:
We say that $\mathcal{P}^{a:b}$ ($\mathcal{P}^{[a:b]}$) is a best fractional (circular) $\mathcal{P}$-bound for $\mathcal{Q}$ if $\mathcal{Q} \subset \mathcal{P}^{a:b}$ ($\mathcal{P}^{[a:b]}$) but $\mathcal{Q} \not\subset \mathcal{P}^{r:s}$ ($\mathcal{P}^{[r:s]}$) for any $(r : s) < (a : b)$ (as rational numbers).

Problem 28.1. *Determine best fractional and circular bounds of planar graphs.*

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29 WITNESSES FOR H-REDUCIBILITY OF INDUCED HEREDITARY GRAPH PROPERTIES

(Ewa Drgas-Burchardt)

For given graphs $G_1, \dots, G_n$ and a graph H with $V(H) = \{v_1, \dots, v_n\}$, we will use the symbol $H[G_1, \dots, G_n]$ to denote the graph whose vertex set is the union of $V(G_1), \dots, V(G_n)$ and whose edge set consists of the union of $E(G_1), \dots, E(G_n)$ with the additional edge set $\{\{x, y\} : x \in V(G_i), y \in V(G_j), \{v_i, v_j\} \in E(H)\}$.
A graph property is any class of graphs that is closed under isomorphisms. A graph property $\mathcal{P}$ is induced hereditary if it is closed under taking induced subgraphs. The class of all induced hereditary graph properties will be denoted by $\mathbf{L}_{\leqslant}$.
Let H be any given graph on vertices $v_1, \dots, v_n$, $n \geqslant 2$. A graph property $\mathcal{P}$ is H-reducible over $\mathbf{L}_{\leqslant}$ if there exist $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathbf{L}_{\leqslant}$ such that $\mathcal{P}$ consists of all graphs whose vertex sets can be partitioned into n parts, possibly empty, satisfying:

1. for each i the graph induced by the i^{th} non-empty partition part is in $\mathcal{P}_i$, and
2. for each i and j with $i \neq j$ there are no edges between the i^{th} and j^{th} parts if v_i and v_j are non-adjacent vertices in H, and
3. there exists a graph $G \in \mathcal{P}$ such that each partition $(V_1, \dots, V_n)$ of G satisfying the above two conditions has the property $V_i \neq \emptyset$ for all $i \in [n]$.

If $\mathcal{P} \in \mathbf{L}_{\leqslant}$ is H-reducible over $\mathbf{L}_{\leqslant}$ then the graph G satisfying the $(3)^{th}$ condition of the definition is called the witness for H-reducibility of $\mathcal{P}$.

Question 29.1. *Let* H *be at least* 2*-vertex graph and let* $\mathcal{P}$ *be the property which is* H*-reducible over* $\mathbf{L}_{\leqslant}$.

1. *Does there exist a witness* $G = H[G_1, \dots, G_n]$ *for* H*-reducibility of* $\mathcal{P}$*?*
2. *If the first question has negative answer in general, then characterize graphs* H *for which such a witness exists.*

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30 Can planar graphs be characterized in terms of the vertex labellings they admit?

(Izak Broere, Erika Raubenheimer)

The basic idea of this contribution is stimulated by a characterization of outerplanar graphs in terms of the vertex labellings they admit as described in [1]. An **outer-planar graph** is a graph G which can be embedded in the plane in such a way that all the vertices of G lie on the boundary of some region, usually chosen to be the exterior region. A graph G of order p is called **cyclic labelable** in [1] if there exists a labelling of the vertices of G with $v_1, v_2, \ldots, v_p$, called a **cyclic labelling**, such that for every $k \geqslant 3$ every cycle of G of length k can be described in terms of its labels as $v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1}$ with $i_1 < i_2 < \cdots < i_k$.

Theorem 30.1. (*[1] and [2]*) *Let* G *be any graph. Then the following conditions on* G *are equivalent:*
(*a*) G *is outerplanar*
(*b*) G *contains no subgraph homeomorphic from* K_4 *or* $K_{2,3}$
(*c*) G *admits a cyclic labelling.*

In [1] a graph G is called **near cyclic labelable** if there exists a labelling of G such that every cycle of G can be described in terms of its labels as $v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1}$ with $i_1 < i_2 < \cdots < i_{j-1} < i_{j+1} < \cdots < i_k$ for some $j \in \{1, 2, \ldots, k\}$. It is then shown that every near cyclic labelable graph is planar. However, not every planar graph is near cyclic labelable - the unique 4-regular graph of order six is an example which proves this remark. In order to pursue this question further we now relax the conditions in the definition of near cyclic labelable further. For this we say, for three positive integers p, q and r that p is **flanked** by q and r if $q > p < r$. We now define a graph G to be n**-defective cyclic labelable** if there exists a labelling of G such that every cycle of G can be described in terms of its labels in such a way that at most n indices of the vertex labels are flanked by the indices of the labels of the two vertices which are adjacent to them in the cycle. However, we then still have

Theorem 30.2. *For every natural number* n *there is a planar graph* G_n *of order* $27n + 2$ *such that for every labelling of* G_n *there is a cycle which, in terms of this labelling, has at least* $n + 1$ *vertices of which the indices of the labels are flanked by the indices of the labels of the two vertices which are adjacent to them in the cycle.*

Hence, for every n, there is a planar graph which is not n-defective cyclic labelable. The problem is therefore to find a type of labelling which characterizes the planarity of a graph.

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31 PATH-SECURITY NUMBER OF A GRAPH

(J. Katrenič, G. Semanišin)

Let G be a graph and $k \geqslant 2$ an arbitrary, fixed integer. Then $S \subset V(G)$ is the k-*path security set* of G if every path on k vertices in G contains a vertex from S. The k-path security number $\psi_k(G)$ of G is the cardinality of a minimum k-path security set in G.
This graph invariant was recently introduced in [2], where the motivation for this problem arises in ensuring data integrity communication in wireless sensor networks.

Theorem 31.1. *For a graph* G *with* n *vertices and* m *edges* $\Psi_3(G) \leqslant \frac{2n+m}{6}$.

Moreover, for an arbitrary rational number x, $1 \leqslant x \leqslant 2$, one can construct a (not-connected) graph G with n vertices and m edges such that $\frac{m}{n} = x$ and $\Psi_3(G) \geqslant \frac{2n+m}{6}$.

Problem 31.1. *Let* G *be a connected graph with* n *vertices and* m *edges. Find a tight upper bound for* $\Psi_k(n, m)$.

Note that $\Psi_2(G)$ corresponds to the size of the minimum vertex cover. Tight upper bounds for Ψ_3 are already known for trees, outerplanar, cubic graphs and hypercubes.

For any fixed integer $k \geqslant 2$ the k-Path Security Problem is NP-hard ([1]). A trivial algorithm to compute Ψ_k runs in time $O(2^n n^{O(1)})$.

Proposition 31.1. *The value of* Ψ_3 *can be computed in time* $O(1.749^{|V(G)|})$.

For trees there is a polynomial time algorithm. For outerplanar graphs, the value of Ψ_3 can be computed in polynomial time. A k-approximation algorithm for Ψ_k is trivial.

Proposition 31.2. *There is a polynomial* 2.25*-approximation algorithm for* Ψ_3.

Problem 31.2. *Is there a polynomial time constant factor approximation for* Ψ_k*?*

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